Variations on a theme : The sum of equal powers of natural numbers (I)

Arkady Alt

This is the first of a series of notes, mostly dedicated to the following problem :

Find, in closed form, the sum

$$S_p(n) := \sum_{k=1}^{n} k^p = 1^p + 2^p + ... + n^p \text{ where } p, n \in \mathbb{N}.$$

These notes can be seen as a continuation of the article "Sums of equal powers of natural numbers" by V. S. Abramovich, published in ***Crux*** Vol. 40 (6).

Let us first consider the simplest special cases of $S_p(n)$ for $p = 1, 2, 3$, testing various approaches to find the most suitable way for consideration of the general case.

1 Finding $S_1(n) := 1 + 2 + 3 + \cdots + n$

We reproduce the way that young K.F. Gauss apocryphally solved the problem at the age of 10, by grouping terms. Since

$$S_1(n) = \sum_{k=1}^{n} k = \sum_{k=1}^{n} (n - k + 1)$$

we have

$$2S_1(n) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} (n - k + 1) = \sum_{k=1}^{n} (k + (n - k + 1)) = \sum_{k=1}^{n} (n + 1) = n(n + 1)$$

and thus

$$S_1(n) = \frac{n(n + 1)}{2}. \tag{1}$$

We can also solve it by the method popularly known as *telescoping*, or the *difference method of summation.* Suppose that it is possible to find a sequence $b_1, b_2, ..., b_k, ...$ such that $a_k = b_{k+1} - b_k,\ k = 1, 2, ...,$. Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_{k+1} - b_k) = b_{n+1} - b_1\ . \tag{2}$$

This is justified in Appendix 1.

Since $k^2 - (k-1)^2 = 2k - 1$, then

$$\begin{aligned}
& \sum_{k=1}^{n} \left(k^2 - (k-1)^2\right) = \sum_{k=1}^{n} (2k-1) \\
\Rightarrow \quad & n^2 - (1-1)^2 = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \quad \text{(telescoping the left hand side)} \\
\Rightarrow \quad & n^2 = 2\sum_{k=1}^{n} k - n \\
\Rightarrow \quad & \sum_{k=1}^{n} k = \frac{n(n+1)}{2}\ .
\end{aligned}$$

Exercise 1 *Prove the same identity starting with the observation that*

$$k^2 - (k-1)^2 + 1 = 2k.$$

2 Finding $S_2(n) := 1^2 + 2^2 + 3^2 + \cdots + n^2$

We can find this by analogy with the previous example, by telescoping sums of cubes and making use of our knowledge of S_1 and S_0. Since $k^3 - (k-1)^3 = 3k^2 - 3k + 1$, we have

$$\begin{aligned}
& \sum_{k=1}^{n} \left(k^3 - (k-1)^3\right) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
\Rightarrow \quad & n^3 = 3S_2(n) - 3S_1(n) + n \\
\Rightarrow \quad & 3S_2(n) = n^3 - n + 3S_1(n) \\
\Rightarrow \quad & 3S_2(n) = n^3 - n + \frac{3n(n+1)}{2} = \frac{n(n+1)(2n+1)}{2} \\
\Rightarrow \quad & S_2(n) = \frac{n(n+1)(2n+1)}{6}\ . \qquad (3)
\end{aligned}$$

Noting that

$$(k+2)(k+1)k - (k+1)k(k-1) = 3k(k+1),$$

we get

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1) &= \frac{1}{3}\sum_{k=1}^{n} \left((k+2)(k+1)k - (k+1)k(k-1)\right) \\
&= \frac{n(n+1)(n+2)}{3}
\end{aligned}$$

and, using the identity $k^2 = k\,(k+1) - k$, we obtain

$$S_2\,(n) = \sum_{k=1}^{n} k\,(k+1) - \sum_{k=1}^{n} k = \frac{n\,(n+1)\,(n+2)}{3} - \frac{n\,(n+1)}{2} = \frac{n\,(n+1)\,(n+2)}{6}\ .$$

This method may appear non-obvious, but the trick is one that is often used in combinatorics and probability theory, and well worth learning. The heart of it is that the *falling factorials*

$$(k)_n := k(k-1)\cdots(k-n+1)$$

and *rising factorials*

$$k^{(n)} := k(k+1)\cdots(k+n-1)$$

play better with their neighbors than powers do. In particular,

$$(k)_n - (k-1)_n = n\cdot(k-1)_{n-1} \text{ and } k^{(n)} - (k-1)^{(n)} = n\cdot k^{(n-1)}\ . \qquad (4)$$

The case $n = 3$ of the first identity is what we started with.

Exercise 2 *Prove* (4)*.*

Exercise 3 *Use* (4) *and* (2) *to show that*

$$\sum_{k=1}^{n} k^{(m)} = \frac{n^{(m+1)}}{m+1}.$$

3 Finding $S_3(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$

Exercise 4 *Note that* $k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$ *. Use this, and the methods of subsection* 2*, to evaluate* $S_3(n)$*.*

This time we will use rising factorials. First, we will find a representation of k^3 in the form

$$k^3 = a + bk + ck\,(k+1) + k\,(k+1)\,(k+2)\ .$$

This can be done by expanding and equating like terms, but (as when finding partial fractions expansions without repeated roots) it is easier to evaluate by plugging in values that make one or more summands vanish. By substituting $k = 0, -1, -2$ in the equation above, we obtain $a = 0, a - b = -1, a - 2b + 2c = -8$, whence $a = 0, b = 1, c = -3$. Since

$$k^3 = k - 3k\,(k+1) + k\,(k+1)\,(k+2)\ ,$$

we have

$$S_3\,(n) = \sum_{k=1}^{n} k - 3\sum_{k=1}^{n} k\,(k+1) + \sum_{k=1}^{n} k\,(k+1)\,(k+2)\,.$$

We already know the closed forms of $\sum_{k=1}^{n} k\,(k+1)$ and $\sum_{k=1}^{n} k$. By Exercise 3, we have

$$\sum_{k=1}^{n} k\,(k+1)\,(k+2) = \frac{n\,(n+1)\,(n+2)\,(n+3)}{4}$$

and, therefore,

$$\begin{aligned} S_3\,(n) &= \frac{n\,(n+1)}{2} - n\,(n+1)\,(n+2) + \frac{n\,(n+1)\,(n+2)\,(n+3)}{4} \\ &= \frac{n^2\,(n+1)^2}{4}\,. \end{aligned} \tag{5}$$

Exercise 5 *Do the same using falling factorials : that is, find* $S_3\,(n)$ *by setting*

$$k^3 = a + bk + ck\,(k-1) + k\,(k-1)\,(k-2)$$

and finding the appropriate sums.

Exercise 6 *Prove the identity*

$$\frac{k^2\,(k+1)^2}{4} - \frac{(k-1)^2\,k^2}{4} = k^3$$

and use it together with (2) *to obtain the closed form for* $S_3\,(n)$*.*

4 Recurrences For The General Case

Modifying the ideas of sections 1 and 2, we obtain

$$\begin{aligned} & (k+1)^{p+1} - k^p = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i} \\ \Rightarrow\quad & \sum_{k=1}^{n} \left((k+1)^{p+1} - k^p\right) = \sum_{k=1}^{n}\sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i} \\ \Rightarrow\quad & (n+1)^{p+1} - 1^p = \sum_{i=1}^{p+1} \binom{p+1}{i} \sum_{k=1}^{n} k^{p+1-i} \\ \Rightarrow\quad & (n+1)^{p+1} - 1 = \sum_{i=1}^{p+1} \binom{p+1}{i} S_{p+1-i}\,(n) \\ \Rightarrow\quad & (n+1)^{p+1} - 1 = (p+1)\,S_p\,(n) + \sum_{i=2}^{p} \binom{p+1}{i} S_{p+1-i}\,(n) + S_0\,(n) \\ \Rightarrow\quad & S_p\,(n) = \frac{(n+1)^{p+1} - 1 - \sum_{k=0}^{p-1} \binom{p+1}{k} S_k\,(n)}{p+1} \end{aligned} \tag{6}$$

for all $p \in \mathbb{N}$, letting $S_0(n) := n$.

Exercise 7 *For any $p \in \mathbb{N}$, show that*

$$k^{p+1} - (k-1)^{p+1} = \sum_{i=1}^{p+1} \binom{p+1}{i} k^{p+1-i} .$$

Now, sum left and right sides for $k = 1, \ldots, n$ and thus show that

$$S_p(n) = \frac{n^{p+1} + \sum_{k=0}^{p-1} (-1)^{p-k+1} \binom{p+1}{k} S_k(n)}{p+1} \tag{7}$$

for $p \in \mathbb{N}$.

So we have recursive representations of $S_p(n)$ in terms of $S_{p-1}(n), \ldots, S_1(n)$. An easy corollary of this result is the fact that $S_p(n)$ is polynomial of degree $p+1$ in n (use mathematical induction on (6).)

5 Appendix

Why does the difference method work? Suppose that $a_k = b_{k+1} - b_k,\ k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} (b_{k+1} - b_k) = \sum_{k=1}^{n} b_{k+1} - \sum_{k=1}^{n} b_k = \sum_{k=2}^{n+1} b_k - \sum_{k=1}^{n} b_k = b_{n+1} - b_1.$$

Informally,

$$a_1 + a_2 + a_3 + \cdots + a_n = (b_2 - b_1) + (b_3 - b_2) + (b_4 - b_3) + \cdots + (b_{n+1} - b_n)$$

and all terms except $-b_1$ and b_{n+1} cancel.

Arkady Alt
San Jose, CA, USA